Lab 1: Butterworth Filters

UC Davis Physics 116B
Rev 1/4/2019

Introduction

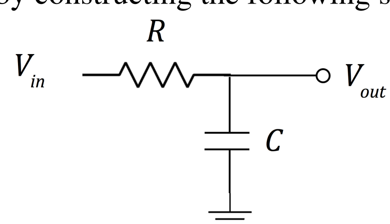
As we discussed in class, Butterworth filters are a class of filters characterized by a transfer function with a magnitude

$$|g(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^{2n}}}$$

where f_0 is the 3dB cutoff frequency and n is the order of the filter. In this lab, you will build and test first, second, and third order Butterworth filters.

First Order Butterworth Filter (Low-pass RC Filter)

A first order Butterworth filter is just a low pass RC filter, so for this section, you will simply repeat a measurement from last quarter by constructing the following simple circuit.



Use $R = 1.5\text{k}\Omega$ and $C = .01\mu\text{F}$ (10nF). For your write-up, show that in s-space, the transfer function is given by

$$g(s) = \frac{1}{1 + \left(\frac{s}{\omega_0}\right)}$$

where $\omega_0 = 1/(RC)$, and that the magnitude of the frequency response is therefore

$$|g(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^{2}}}$$

where $f_0 = 1/(2\pi RC)$. Drive the circuit with a sine wave from the function generator, with an amplitude of 10V_{pp}.

Calculate the -3dB frequency f_0 [Hz], then measure it by finding the frequency at which the peak-to-peak voltage falls to $1/\sqrt{2}$ of its low frequency value. It should be within 5-10% of your calculation, given the precision of the parts we're using.

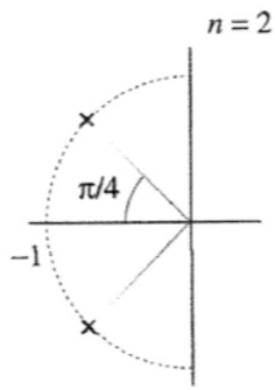
Once you have determined f_0, then hold the input amplitude fixed and measure the output amplitude for the following frequencies:

$$f = \left[\frac{f_0}{10}, \frac{f_0}{2}, f_0, 2f_0, 10f_0\right]$$

Use the cursors on the oscilloscope to get as accurate a measurement of the peak-to-peak voltage as you can at each frequency. As a quick check, recall that we showed in class that the gain at $f = 10f_0$ should be $-20n$ dB, where n is the order of the Butterworth filter. Is it?

Second Order Butterworth Filter

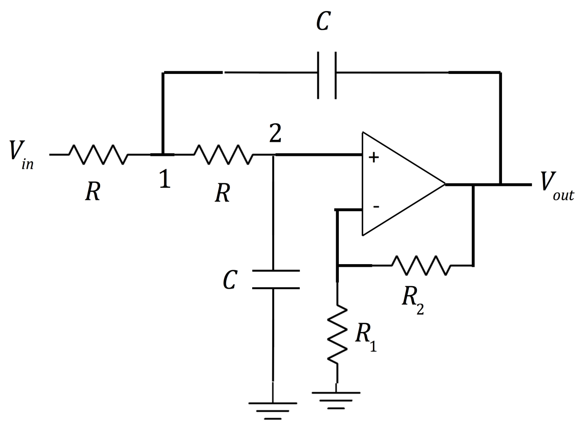
We showed in class that a second-order Butterworth filter should have the following poles in the s-plane



which give a transfer function of the form

$$g(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

As shown in the Appendix (and in class), the following circuit



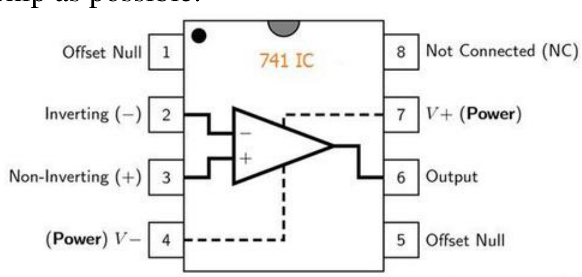
has the transfer function

$$g(s) = \frac{A}{\left(\frac{s}{\omega_0}\right)^2 + (3 - A)\left(\frac{s}{\omega_0}\right) + 1}$$

where $\omega_0 = 1/(RC)$ and A is the gain of the op amp circuit, given by

$$A = 1 + \frac{R_2}{R_1}$$

Construct the circuit using the LM741 op amp, and the same R and C values from the first section. The pin-outs for the LM741 are shown below. Use +15V and -15V for the power levels. If you observe any high frequency oscillations, you may need to put .1μF capacitors from the power pins to ground, as close to the chip as possible.



Use a 3.9 kΩ resistor for R_1, and pick a value of R_2 to make the coefficient of s as close to $\sqrt{2}$ as possible (i.e set the gain to $A = 3 - \sqrt{2}$).

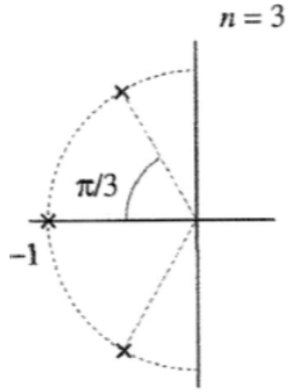
Use a very low frequency signal ($\sim$10 Hz) to verify that A has the correct value. As in the first section, measure f_0 by finding the frequency at with the amplitude falls to the low frequency value divided by $\sqrt{2}$. Once you have done this, measure the gain over the same range of frequencies as in the first section.

Note, unlike the passive filter, there will be an overall gain to this circuit, so decibels are calculated using $|g(f)|/A|$, where A is your *measured* value. Verify that the magnitude of the response function has the form

$$|g(f)| \propto \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^4}}$$

Third Order Butterworth Filter

A third order Butterworth filter has three poles in the s-plane



and is described by a transfer function of the form

$$g(s) = \frac{1}{(s^2 + s + 1)(1 + s)}$$

Construct this filter by adding the first order RC filter of the first section to the second order filter from the second section. Replace R_2 with a value that will give the correct coefficient for s in the quadratic term (hint: set $A = 3 - 1$). Put the first order filter *after* the second order filter (why?).

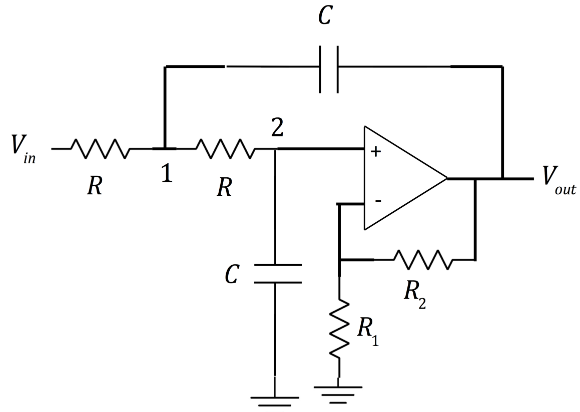
Repeat the measurements you made in the second section, and verify that the magnitude of the response function has the form

$$|g(s)| \propto \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^6}}$$

For your lab report, plot the response of all three filters on the same plot as dB vs. f/f_0, where f_0 is your measured value for the -3dB frequency in each case. Use a log scale for the frequency axis.

Appendix: Second Order Active FIlter

To calculate the transfer function of the following circuit



we apply the current sum rules at points 1 and 2

$$\frac{V_{in} - V_1}{R} + \frac{V_{out} - V_1}{\frac{1}{sC}} + \frac{V_2 - V_1}{R} = 0 \rightarrow V_{in} - (2 + sRC)V_1 + V_2 + sRCV_{out} = 0$$

$$\frac{V_2 - V_1}{R} + \frac{V_2}{\frac{1}{sC}} = 0 \rightarrow V_1 = (1 + sRC)V_2$$

Substituting $(1 + sRC)V_2$ for V_1 in the first equation gives

$$V_{in} = \left[(2 + sRC)(1 + sRC) - 1\right] V_2 - sRCV_{out}$$

We now recognize that the final stage is just a non-inverting op-amp circuit, with a gain of $A = (1 + R_2/R_1)$, so we substitute V_{out}/A for V_2, and expand the polynomial to get

$$V_{out} = \frac{A}{(sRC)^2 + (3 - A)sRC + 1} V_{in}$$

If we define a normalized variable

$$\tilde{s} \equiv \frac{s}{\omega_0}$$

where $\omega_0 = 1/(RC)$, then this becomes

$$V_{out} = \frac{A}{\tilde{s}^2 + (3 - A)\tilde{s} + 1} V_{in}$$

This circuit is a powerful tool in building filters, because we can tune then the linear coefficient of s by simply adjusting the gain of the last stage, thereby placing a pair of complex poles wherever we want.

In the particular case of Butterworth filters, the real part of the complex pole pairs always lies between -1 and 0, meaning the linear coefficient in the denominator will be between 0 and 2, so A will always be between 1 and 3. Such a gain can always be generated by a value of R_2 between 0 and $2R_1$. Thus, we can use a series of these circuits to build a Butterworth filter to any order, with odd orders requiring the addition of a single RC stage to generate the real pole.